CG2023 Final Examination Cheat-sheet

1. Fourier series in trigonometric form:

$$x_p(t) = a_0 + 2 \cdot \sum_{k=1}^{\infty} (a_k \cos(\frac{2\pi k}{T_p}) + b_k \sin(\frac{2\pi k}{T_p}))$$

where $a_k = (c_{-k} + c_k)/2$ and $b_k = (c_{-k} - c_k)/2$

2. Spectral properties of a real signal:

1) $x(t)$ is <u>real</u>, we have $X^*(f) = X(-f)$, which leads to
$$|X(f)| = |X(-f)| \text{ and } \angle X(f) = -\angle X(-f)$$
2) $x(t)$ is <u>real and even</u>, we have $X(f)$ is also real and even
$$X^*(f) = X(f) \text{ and } X(f) = X(-f)$$
3) $x(t)$ is <u>real and odd</u>, we have $X(f)$ is imaginary and odd
$$X^*(f) = -X(f) \text{ and } X(f) = -X(-f)$$

3. Properties of the Dirac-δ function:

1) *Symmetry*: $\delta(t) = \delta(-t)$
2) *Sampling*: $x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)$
3) *Sifting*: $\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt = x(\lambda)$
4) *Replication*: $x(t) * \delta(t-\xi) = x(t-\xi)$
5) *White spectrum*: $\Im\{\delta(t)\} = \Im^{-1}\{\delta(t)\} = 1$

4. Use Fourier transform on periodic signals:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} = \sum_{k=-\infty}^{\infty} c_k \cdot \delta(f - kf_p)$$

5. Fourier transform of the generating function:

$$x_p(t) = g(t) * \sum_n \delta(t - nT_p) = \sum_n g(t - nT_p)$$

$$\rightleftharpoons X_p(f) = \sum_k f_p G(kf_p)\delta(f - kf_p) \text{ thus } c_k = f_p G(kf_p)$$

6. DC value (or called* average value*):

1) Original definition: $c_0 = \lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\tau}^{\tau} x(t)dt = \frac{X(0)}{\infty}$.
2) For periodic signal: $c_0 = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x(t)dt$.
3) In the frequency domain: $\frac{X(0)}{\infty} = \frac{\int_{-\infty}^{\infty} x(t)dt}{\infty} = c_0$.
4) The DC value of a signal is K if $X(f)$ contains a $K\delta(f)$ term.
5) *All bounded energy signals have <u>zero DC value</u>.*
6) *All bounded periodic signals are <u>power signals</u>.*

7. Energy & energy spectral density:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} E_x(f)dt$$

8. Power & power spectral density:

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T}|X_T(f)|^2 df = \int_{-\infty}^{\infty} P_x(f)df$$

Furthermore, for periodic signals, we have

$$P_x(f) = \sum_k |c_k|^2 \delta(f - kf_p) \text{ and } P = \sum_k |c_k|^2$$

9. Magnitude & phase of a complex number:

1) Any complex number can be expressed as $X = a + \boldsymbol{j}b = |X| \cdot e^{j\angle X}$, whose magnitude is $|X| = \sqrt{X \cdot X^*} = \sqrt{a^2 + b^2}$, and whose phase can be written as $\angle X = \frac{Im\{X\}}{Re\{X\}} = \frac{b}{a}$.
2) In the form of $\frac{1}{a+\boldsymbol{j}b}$, its magnitude is $\frac{1}{\sqrt{a^2+b^2}}$, and its phase is $-tan^{-1}(\frac{b}{a})$.

10. Properties of an LTI (linear time-invariant) system:

1) Impulse response (in t-domain): $y(t) = x(t) * h(t)$.
2) Frequency response (in f-domain): $Y(f) = X(f) \cdot H(f)$.
3) With transfer function $\widetilde{H}(\boldsymbol{j}\omega)$, if input is $x(t) = A\cos(\omega t + \varphi)$, we have output $y(t) = A\left|\widetilde{H}(\boldsymbol{j}\omega)\right|\cos(\omega t + \varphi + \angle\widetilde{H}(\boldsymbol{j}\omega))$.

11. Stability of an LTI system with poles and/or zeros:

1) <u>BIBO stable</u>: All poles lie on the left-half s-plane, i.e. $Re[s] < 0$.
2) <u>Marginally stable</u>: One or more poles lie on the $\boldsymbol{j}\omega$ axis, the else on the left-half s-plane.
3) <u>Unstable</u>: One or more holes on the right-half s-plane, or one or more repeated holes the $\boldsymbol{j}\omega$ axis

12. DC gain of a N[th]-order LTI system:

1) If there are more differentiators, DC gain is $\widetilde{H}(0) = 0$.
2) If there are more integrators, DC gain is $\widetilde{H}(0) = \infty$.
3) *DC gain should not be expressed in dB, although dB is used for gain.*

13. Asymptotic value of phase response:

1) High-frequency: $\lim_{\omega\to\infty} \angle\widetilde{H}(\boldsymbol{j}\omega) = [\#\text{poles} - \#\text{zeros}] \times (-90°)$
2) Low-frequency: $\lim_{\omega\to 0} \angle\widetilde{H}(\boldsymbol{j}\omega) = [\#\text{integrators} - \#\text{differentiators}] \times (-90°)$

14. Asymptotic value of magnitude response:

1) High-frequency: $\lim_{\omega\to\infty} |\widetilde{H}(j\omega)|_{dB} = [\#\text{poles} - \#\text{zeros}] \times (-20dB/decade)$

2) Low-frequency: $\lim_{\omega\to\infty} |H^{\sim}(j\omega)|_{dB} = [\text{\#integrators} - \text{\#differentiators}] \times (-20)$
3) *Notice*: The number of poles/zeros here *includes* the number of integrators/differentiators.

15. Resonance in 2nd-order systems

Given $\widetilde{H}(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, resonance happens when $\zeta < 1/\sqrt{2}$.

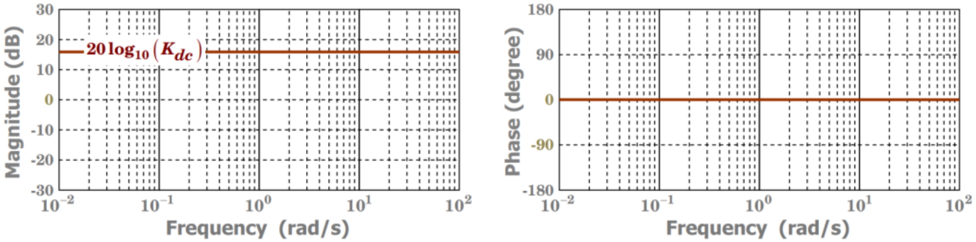
1) *Resonant frequency*: $\omega_r = \omega_n\sqrt{1-2\zeta^2}$
2) *Resonant peak*: $M_r = \left|\widetilde{H}(\boldsymbol{j}\omega_r)\right| = \frac{K}{2\zeta\sqrt{1-\zeta^2}}$

16. Sampling below Nyquist rate *for bandpass signals*

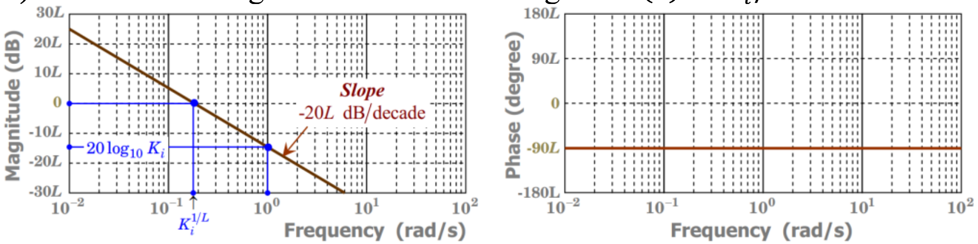
1) *Perfectly-overlapping images*: $\boldsymbol{f_s} = \frac{2f_c}{k}$, where $k = 1, 2, \dots, \left\lfloor\frac{2f_c}{B}\right\rfloor$.
2) *Un-aliased spectral images*: $\frac{2f_c+B}{k+1} \le \boldsymbol{f_s} \le \frac{2f_c-B}{k}$, $k = 1, 2, \dots, \left\lfloor\frac{2f_c-B}{2B}\right\rfloor$.
3) The above two are only possible when $\boldsymbol{f_c} \ge \boldsymbol{B}$.
4) *Notice*: for perfectly-overlapping images, a **division-by-2** is necessary on the reconstruction filter.

17. Drawing of straight-line Bode plots:

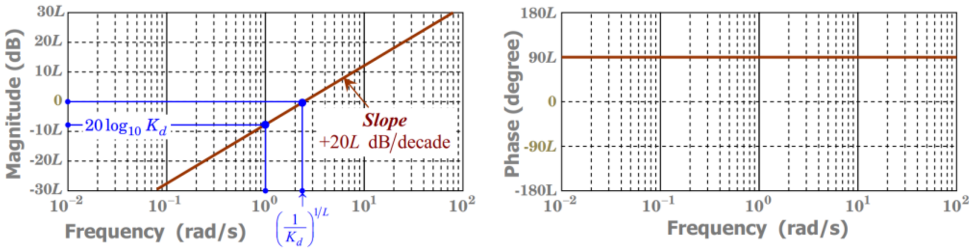
1) Constant $\widetilde{H}(s) = K_{dc}$:



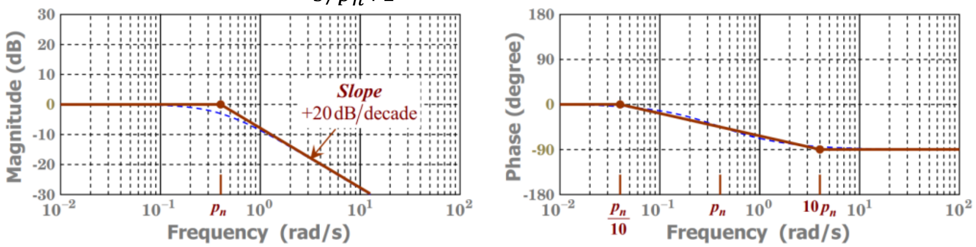
2) L cascaded integrators with combined gain $\widetilde{H}(s) = K_i/s^L$:



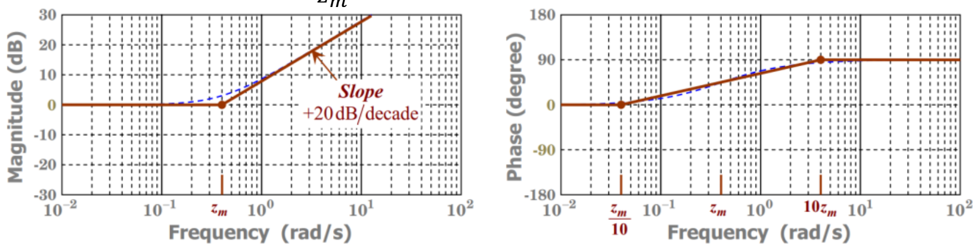
3) L cascaded differentiators with combined gain $\widetilde{H}(s) = K_d \cdot s^L$:



4) Pole factor $\widetilde{H}(s) = \frac{1}{s/p_n+1}$ with $\widetilde{H}(0) = 1$:



5) Zero factor $\widetilde{H}(s) = \frac{s}{z_m} + 1$ with $\widetilde{H}(0) = 1$:



6) 2nd-order factor $\widetilde{H}(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ with $\widetilde{H}(0) = 1$:

a) When $\zeta > 1$ (***over-damped***), treat as two cascaded pole factors;
b) When $\zeta = 1$ (***critical-damped***), treat as two repeated pole factors;
c) When $\zeta < 1$ (***under-damped***), approximate as $\zeta = 1$.

For 2nd and 3rd case, draw the straight-line Bode plot as follows: