CG2023 Midterm Cheat-sheet

1. Fourier series in exponential form:

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}$$

2. Fourier series in trigonometric form:

$$x_p(t) = a_0 + 2 \cdot \sum_{k=1}^{\infty} (a_k \cos(\frac{2\pi k}{T_p}) + b_k \sin(\frac{2\pi k}{T_p}))$$

where $a_k = (c_{-k} + c_k)/2$ and $b_k = (c_{-k} - c_k)/2$

3. Spectral properties of a real signal:

1) $x(t)$ is real, we have $X^*(f) = X(-f)$, which leads to

$$|X(f)| = |X(-f)| \text{ and } \angle X(f) = -\angle X(-f)$$

2) $x(t)$ is real and even, we have $X(f)$ is also real and even

$$X^*(f) = X(f) \text{ and } X(f) = X(-f)$$

3) $x(t)$ is real and odd, we have $X(f)$ is imaginary and odd

$$X^*(f) = -X(f) \text{ and } X(f) = -X(-f)$$

4. Properties of the Dirac-δ function:

1) Symmetry: $\delta(t) = \delta(-t)$

2) Sampling: $x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)$

3) Sifting: $\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt = x(\lambda)$

4) Replication: $x(t) * \delta(t-\xi) = x(t-\xi)$

5) White spectrum: $\mathfrak{F}\{\delta(t)\} = \mathfrak{F}^{-1}\{\delta(t)\} = 1$

5. Use Fourier transform on periodic signals:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} = \sum_{k=-\infty}^{\infty} c_k \cdot \delta(f - kf_p)$$

6. Fourier transform of the generating function:

$$x_p(t) = g(t) * \sum_n \delta(t - nT_p) = \sum_n g(t - nT_p)$$

$$X_p(f) = \sum_k f_p G(kf_p)\delta(f - kf_p) \text{ thus } c_k = f_p G(kf_p)$$

7. According to Fourier transform, $X(0) = \int_{-\infty}^{\infty} x(t)dt$, we have the dc value of $x(t)$ is $c_0 = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x(t)dt$ (for periodic signals), and it is also $c_0 = \lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\tau}^{\tau} x(t)dt = \frac{X(0)}{\infty}$.

8. Energy & energy spectral density:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} E_x(f) dt$$

9. Power & power spectral density:

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2 df = \int_{-\infty}^{\infty} P_x(f) df$$

Furthermore, for periodic signals, we have

$$P_x(f) = \sum_k |c_k|^2 \delta(f - kf_p) \text{ and } P = \sum_k |c_k|^2$$

$$\textbf{DC value} \text{ of } x(t) = \begin{cases} \frac{1}{T}\int_{-0.5T}^{0.5T} x(t)dt = c_0 & \text{if } x(t) \text{ is periodic with period } T \\ 0 & \text{if } x(t) \text{ is an energy signal} \\ K & \text{if } X(f) \text{ contains a } K\delta(f) \text{ term} \end{cases}$$

AVERAGE POWER of a periodic signal $x(t)$ with period T:

$$P = \underbrace{\frac{1}{T}\int_{-0.5T}^{0.5T} |x(t)|^2 dt}_{\text{time-domain}} = \underbrace{\int_{-\infty}^{\infty} P_x(f) df = \sum_k |c_k|^2}_{\text{frequency-domain}}$$

PSD, $P_x(f)$, is a function of frequency f
Power, P, is **not** a function of frequency f

TOTAL ENERGY of an energy signal $x(t)$:

$$E = \underbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}_{\text{time-domain}} = \underbrace{\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} E_x(f) df}_{\text{frequency-domain}}$$

ESD, $E_x(f)$, is a function of frequency f
Energy, E, is **not** a function of frequency f

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)dt$$

$$X(f) = |X(f)|\exp(j\angle X(f)) \begin{cases} |X(f)| = \sqrt{X(f)X^*(f)} & \cdots \text{ Magnitude Spectrum} \\ \angle X(f) = \tan^{-1}\left(\frac{\text{Im}\{X(f)\}}{\text{Re}\{X(f)\}}\right) & \cdots\cdots\cdots \text{ Phase Spectrum} \end{cases}$$

REPLICATION:

$$\underset{\substack{\uparrow \\ \text{convolution}}}{f(z) * \delta(z - shift)} = f(z - shift) \quad \left\{ \begin{array}{l} \text{Replace the argument of } f(\cdot) \text{ with the argument of} \\ \delta(\cdot) \text{ and remove } \delta(z - shift) \end{array} \right.$$

Example: $Af(z) * \sum_{n=-\infty}^{\infty} B\delta(z - nC) = \sum_{n=-\infty}^{\infty} AB\, f(z) * \delta(z - nC) = \sum_{n=-\infty}^{\infty} AB\, f(z - nC)$

SAMPLING:

$$\underset{\substack{\uparrow \\ \text{multiplication}}}{f(z)\delta(z - shift)} = f(shift)\delta(z - shift) \quad \left\{ \begin{array}{l} \text{Replace the argument of } f(\cdot) \text{ with the value} \\ \text{of } z \text{ that zero out the argument of } \delta(\cdot) \end{array} \right.$$

Example: $Af(z) \sum_{n=-\infty}^{\infty} B\delta(z - nC) = \sum_{n=-\infty}^{\infty} AB\, f(z)\delta(z - nC) = \sum_{n=-\infty}^{\infty} AB\, f(nC)\delta(z - nC)$
