1. Finance & Economics

1) *Rule of 72:* take $72/\alpha$ years to double with compound interest rate $\alpha\%$.
2) *Present value:* $Y = X/(1+\alpha\%)^n$
3) *Geometric series:* $a + ax + ax^2 + \cdots + ax^n = (a - ax^{n+1})/(1-x)$
4) *Mortgage:* amount owned after time $C_k = C \cdot (1+\alpha\%)^k - R \cdot \sum_{j=0}^{k-1}(1+\alpha\%)^j$, solving and get $R = \frac{C\cdot(1+\alpha)^{12n}}{\sum_{j=0}^{12n-1}(1+\alpha)^j} = \frac{C\alpha\cdot(1+\alpha)^{12n}}{(1+\alpha)^{12n}-1} = \frac{C\alpha\cdot(1+l)^n}{(1+l)^n-1}$.
5) Lorenz curve: arrange people in increasing wealth, L(x) = proportion of wealth owned by the lowest proportion x of the population ($0 \le L(x) \le 1$).
6) Gini coefficient: the deviation of Lorenz curve from line of perfect equality.

2. Logic & Set Theory

1) *Implication:* $P \to Q = \neg P \vee Q$
2) *There are infinite number of primes:* otherwise $p_1 p_2 \cdots p_n + 1$.
3) *Functions:* bijective function implies the <u>same cardinality</u>; injective means no two x's point to the same y; surjective means every y is pointed by some x.
4) *Common set cardinality:* $\mathbb{Z}^+ = \mathbb{N} = \mathbb{Z} = \mathbb{Q} < \mathbb{R}, \mathbb{C}$.

3. Social Theory

1) *Properties of a desirable social choice function:*
a. <u>Anonymity:</u> result is the same if two voters exchange the votes;
b. <u>Neutrality:</u> result is reversed if everyone switches his/her vote;
c. <u>Monotonicity:</u> result is the same if some change from loser to winner.
d. <u>Independence of irrelevant alternatives:</u> if the system ranks *A* over *B* in the presence of *C*, *A* should be over *B* without *C*.
e. <u>Unanimity (Pareto condition):</u> if every voter prefers *A* over *B*, then the system should rank *A* over *B* in overall.
2) *Arrow's Theorem:* no voting system can satisfy all 5 properties above.
3) *For a 2-candidate system* to satisfy property a – c, it must be a quota system.
4) *Common voting systems (for 2 candidates):*
a. <u>Majority rule:</u> the one with more votes win; b. <u>Minority rule:</u> the one with fewer votes win; c. <u>Imposed rule:</u> the result is pre-determined; d. <u>Dictatorship:</u> only the dictator's vote counts; e. <u>Jury trail:</u> one can only win if everyone votes for him/her; f. <u>Quota system:</u> one only wins if his/her vote is higher than x%.
4) *Condorcet winner:* beats every other candidate in a head-to-head contest.
a. <u>Condorcet winner</u> may not exist; but if exists, must be unique.
b. <u>Condorcet winner criterion (CWC):</u> a Condorcet winner (if he/she exists) is always the winner in the system.
5) *Common voting systems (for more than 2 candidates):*
a. <u>Plurality rule:</u> the one with highest percentage of votes win;
b. <u>Borda count:</u> the one with the lowest point (1 point for first preference, 2 points for second preference, and so on);
c. <u>Sequential pairwise voting (SPV):</u> put all candidates in a sequence, use the majority rule to let the first two compete, the winner competes with the third one, and so on;
d. <u>Hare system (instant run-off voting):</u> examine the first preference of the voters and eliminate the candidate with lowest votes, update preference lists for all voters and repeat the procedure.
6) *Problems with common voting systems (for > 2 candidates):* plurality rule does not satisfy CWC, Borda count violates independence of irrelevant alternatives, SPV violates unanimity, Hare system violates monotonicity.

4. Game Theory

1) Zero sum, varied sum, saddle point, worst case analysis, dominant strategy.
2) *Nash equilibrium:* in a game with finite players, a strategy profile such that one player cannot improve his/her payoff by changing his/her strategy while others keeping their strategies unchanged.
3) *Nash Theorem:* in a game with finite players and finite strategies, a Nash equilibrium always exists under mixed strategy approach.
4) Mixed Strategy: guarantee a fixed expected payoff **indifferent** of the other party's action.

5. Division Theory

1) *Two properties of fair division:*
a. <u>Proportionality:</u> every player feels he/she receives at least $1/n$ of value;
b. <u>Envy-free</u> (implies proportionality): everyone feels its division $\ge$ others;
c. <u>Equitable:</u> both players receive the same number of points;
d. <u>Pareto-optimal:</u> no other method can make one better without others worse.
2) *Fair division for 2 parties:* A divides and B chooses.
3) *Steinhaus Proportional Procedure (for 3 players):*
a. A divides the value into 3 pieces;
b. B and C approves pieces he/she thinks that is worth at least 1/3 (B and C each approves at least one piece);
c. If B approves X and C approves Y, then $Z \Rightarrow A, X \Rightarrow B, Y \Rightarrow C$;
d. If B and C both only approve X, then $Z \Rightarrow A$ and applies fair division for 2 parties on the combined piece of X and Y.
4) *Banach-Knaster Proportional Procedure (for more than 3 players):*
a. Player 1 cuts a piece he/she thinks that is worth at least 1/n and passes it to the next person; b. If player 2 thinks the piece's value > 1/n, then he/she trims it down to 1/n; otherwise, directly passes it to the next person; c. Repeat the same procedure for player 3 onwards; d. The last person who trims gets it, then start again until finished.
5) *Selfridge-Conway Envy-free Procedure (for 3 players):*

 Good Luck!

a. Player 1 divides into 3 equal parts (in his/her view) and passes to next;
b. Player 2 orders the 3 parts in decreasing order (as A, B, C), and trims off A to get $A' = B$ and $T = A - A'$. Let player 3 choose in A', B, C;
c. Player 2 chooses in A', B and must choose A' if available;
d. Player 1 gets the remaining piece (either B or C);
e. Player 2 divides T into 3 pieces, which are chosen in the order 3, 1, 2.

6) *Adjusted Winner Bidding Procedure (for 2 players):*
a. Allocate items to the person with strictly higher points; allocate tied items to the person with fewest point, one at a time;
b. If the point total is equal, done; otherwise, declare the person with most points as initial winner. For each item, calculate the ratio of the initial winner to loser's points. Order the ratios increasingly, transfer items (or fractions of items) from initial winner to loser until total points equal *(equitable, envy-free and Pareto-optimal)*.

7) *Knaster Inheritance Procedure (method of Sealed Bids, for > 2 players):*
a. Bidding: every player bids by how much each item is worth;
b. Allocation: allocates each item to the one with the highest bid;
c. First settlement: each player receives/pays money to ensure fair share;
d. Division of surplus: divide left money equal among all players.

6. Graph Theory

1) *Northwest Corner Method (to find initial feasible solution):* start from the top-left corner, move right or down until finished.
2) *Minimal Cost Method (to find better feasible solution):* start from cell with lowest cost, move to the cell with lowest cost in the same row/column.
3) *Stepping Stone Method (to improve feasible solution):* follow a stepping stone path to move units to an empty cell.
4) *Modified Distribution Method (to find optimal solution):*
a. Introduce the outsourcing company and find U_i and V_j $(C_{ij} = U_i + V_j)$;
b. Compute $K_{ij} = C_{ij} - (U_i + V_j)$ for all empty cells, if all $K_{ij} \geq 0$, then done; otherwise, find negative cells and apply stepping stone. Repeat until done.
5) *Hungarian Method (for assignment problem):*
a. Subtract the minimal entry for each row and for each column;
b. Draw minimal number of lines to cover all zeros. If the number of lines is n, then done; otherwise, find the minimal item k not covered by any line;
c. Subtract k from each uncovered row, add k to each covered column. Then repeat the above procedure.
6) Eulerian walk is a traversal that uses every edge exactly once, it becomes an Eulerian cycle if the walk starts and ends at the same vertex.
7) Eulerian cycle exists $\Leftrightarrow$ degree of every vertex is even.
8) Eulerian walk exists $\Leftrightarrow$ degree of A and B is odd, every other is even.
9) *Theorem of Konig and Egervary:* for a bipartite graph, the size of minimal vertex cover is equal to the size of a maximum matching.

7. Probability & Statistics

1) *Arrange around table:* $(n-1)!$; *Arrange identical's into boxes:* $C(n+k-1, n)$.
2) *Binomial Theorem:* $(a+b)^n = \sum_{r=0}^{n} C(n,r) \cdot a^{n-r} \cdot b^r$.
3) *Bayes' Theorem:* $P(A|B) = P(B|A) \cdot P(A) / P(B)$.
4) *Strong law of large numbers:* $(X_1 + \cdots + X_n)/n \to \mu$ as $n \to \infty$.
5) *Benford's Law (law of digits):* $P(d) = \log_{10}(d+1) - \log_{10} d$.
6) *Normal Distribution:* $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}$ (68% - 1 σ, 95% - 2 σ, 98.5% - 3 σ).
9) *Central Limit Theorem:* large enough sample tends to become $N(\mu, \sigma/\sqrt{n})$.
10) Sampling distribution: we have $\sigma = \sqrt{p \cdot (1-p)/n}$.

8. Random Walking & Surfing

1) *Gambler's Ruin:* start with *N* and leaves till he gets broken or wins *M* (i.e., leaves with total amount *M* + *N*), each winning with probability *p*.
a. If $p = 1/2$, then probability of eventual winning is $N/(M+N)$;
b. If $p < 1/2$, then let $b = (1-p)/p$ and eventual $(b^N - 1)/(b^{M+N} - 1)$;
c. If $p > 1/2$, then let $b = (1-p)/p$ and eventual $(1 - b^N)/(1 - b^{M+N})$.
2) A vertex is **recurrent** for a random walk if the probability of returning to 0 infinitely often starting from it is 1; **transcient** otherwise.
3) *Polya's Theorem:* for $P(S_n = 0) \sim C \cdot n^{-d/2}$, a simple walk in $\mathbb{Z}^d$ is recurrent if d is 1 or 2; transcient otherwise.
4) Google PageRank: $Rank(A) = (1-d)/N + d \cdot \sum_{B \to A} Rank(B)/|B|$.

9. Coding & Encryption

1) *Universal Product Code (UPC):* (3 * odd digits sum + even digits sum) | 10.
2) *International Standard Book Number (ISBN):* $(\sum_{n=1}^{10} n \cdot a_n) \mid 11$.
3) *Hamming Code:* $sum(s_{1,3,4,5}) \equiv sum(s_{1,2,4,6}) \equiv sum(s_{1,2,3,7}) \equiv 0 \bmod 2$.
4) *QR Code:* version N has $4 \cdot N + 17$ modules.
5) *Fermat's Little Theorem:* for any prime *p*, $a^{p-1} \equiv 1 \, mod \, p$.
6) For any prime *p*, *a* exists one unique multiplicative inverse if they co-prime.
7) There exists a finite field of size *N* if and only if $N = p^r$ for some prime *p*. In this case, such field is unique, although it does not have to be $\mathbb{Z}/p\mathbb{Z}$.
8) *Caesar Cipher:* shift 26 letters all by *k*.
9) *Affine Cipher:* $y \equiv ax + b \bmod N$ where *a* must be prime.
10) *Euler's generalization:* $a^{\varphi(p)} \equiv 1 \bmod p$.

Good Luck!