Fourier Series

- Let *f(x)* be a function with period of 2π, then its Fourier Series is

$$\mathrm{f(x)} = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

- Let *f(x)* be a function with period of 2L, then its Fourier Series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$\text{where } a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\, dx \text{ and } b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\, dx.$$

- Let *f(x)* be an even function with period of 2π, then its Fourier Series is

$$\mathrm{f(x)} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{where } a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \text{ and } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx.$$

(Since $b_n = 0$ for all n) (cosine Fourier Series)

- Let *f(x)* be an even function with period of 2L, then its Fourier Series is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$$\text{where } a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \text{ and } a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\, dx.$$

(Since $b_n = 0$ for all n) (cosine Fourier Series)

- Let *f(x)* be an odd function with period of 2π, then its Fourier Series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx.$

(Since $a_n = 0$ for all n) (sine Fourier Series)

- Let *f(x)* be an odd function with period of 2L, then its Fourier Series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

where $b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx.$

(Since $a_n = 0$ for all n) (sine Fourier Series)

- Important trigonometric identity

$$cos^2 x = \frac{1 + \cos 2x}{2}$$

$$sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin x \cos x = \frac{\sin 2x}{2}$$

$$\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}$$

$$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}$$

$$\cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2}$$

- Usual trigonometric integral

$$\int x \sin(\lambda x)\, dx = \frac{\sin(\lambda x)}{\lambda^2} - \frac{x \cos(\lambda x)}{\lambda}$$

$$\int x^2 \sin(\lambda x)\, dx = \frac{2x \sin(\lambda x)}{\lambda^2} + \frac{(2 - \lambda^2 x^2) \cos(\lambda x)}{\lambda^3}$$

$$\int x \cos(\lambda x)\, dx = \frac{\cos(\lambda x)}{\lambda^2} + \frac{x \sin(\lambda x)}{\lambda}$$

$$\int x^2 \cos(\lambda x)\, dx = \frac{2x \cos(\lambda x)}{\lambda^2} + \frac{(\lambda^2 x^2 - 2) \sin(\lambda x)}{\lambda^3}$$

Functions of multiple variables

- Second derivative test

Assume that function $f(x, y)$ & its 1st & 2nd partial derivatives are continuous in a region containing (a, b), such that $f_x(a, b) = 0$ & $f_y(a, b) = 0$, then let

$$D = f_{xx}(a, b) \cdot f_{yy}(a, b) - f_{xy}(a, b)^2, \text{ so that}$$

(a) If $\boldsymbol{D} > 0$ and $f_{xx}(a, b) > 0$, then $\boldsymbol{f}$ has a **local minimum** at (a, b);
(b) If $\boldsymbol{D} > 0$ and $f_{xx}(a, b) < 0$, then $\boldsymbol{f}$ has a **local maximum** at (a, b);
(c) If $\boldsymbol{D} < 0$, then f has a **saddle point** at (a, b);
(d) If $\boldsymbol{D} = 0$, then **no conclusion** can be drawn.

- Lagrange Multipliers

In order to get the extreme value under certain constraint, we have

$$\mathrm{F}(x, y, z) = f(x, y, z) - \lambda \cdot g(x, y, z)$$

The solution (x, y, z) of equations $F_x = F_y = F_z = F_\lambda = 0$ will help us find the extreme values.

Line integral

- For scalar function

$$\int_c f(x, y)\, ds = \int_a^b f\big(x(t), y(t)\big) \|r'(t)\|\, dt$$

$$\int_c f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \|r'(t)\|\, dt$$

For a curve given as $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, its arc length is

$$s(t) = \int_a^b \|r'(t)\| dt$$

In addition, the shortest distance from a point S (x_0, y_0, z_0) to a plane whose equation is $\Pi: ax + by + cz = d$, is given by

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

- For vector field

$$\int_C \mathbf{F} d\mathbf{r} = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) dt$$

In its component form, it can be written as

$$\int_C \mathbf{F} d\mathbf{r} = \int_C P dx + Q dy + R dz$$

- Fundamental Theorem for line integral

$$\int_C \nabla f d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

where $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$

- Green's Theorem

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Surface integral

- For scalar function

$$\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))\|\mathbf{r}_u \times \mathbf{r}_v\| dA$$

For a surface given as $z = f(x,y)$, its area is the integral of

$$dS = \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}\,dx\,dy$$

- For vector field

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

- Two properties

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$\text{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- Stokes' Theorem

$$\int_C \mathbf{F} d\mathbf{r} = \iint_S (\text{curl}\mathbf{F}) \cdot d\mathbf{S}$$

- Divergence Theorem

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\text{div}\mathbf{F}) dV$$