MA1506 Final Cheat-sheet
AY2016/2017 Semester 2

Part 1 Mathematical Modelling

1. Harmonic Oscillator

0) Newton's 2nd Law tells x as position, $\dot{x}$ as velocity, $\ddot{x}$ as acceleration.

1) *Simple harmonic oscillator* (*pendulum*): We have $m\ddot{x} = -kx$, which is usually written as $\ddot{x} + \omega^2 x = 0$. The phase-amplitude form is $x(t) = A\cos(\omega t - \delta)$. Here, A is amplitude, $T = 2\pi\sqrt{m/k}$ is period, $f = 1/T$ is frequency, ω is angular frequency. When initial conditions are given, we have $A = \sqrt{x_0^2 + (v_0/\omega)^2}$.

2) *Damped harmonic oscillator*: We have $m\ddot{x} + b\dot{x} + kx = 0$, where $m, b, k > 0$. When $b^2 - 4mk > 0$, over dumping, goes to 0 rapidly without oscillation; when $b^2 - 4mk = 0$, critical dumping, goes to 0 rapidly without oscillation; when $b^2 - 4mk < 0$, under dumping, goes to 0 slowly with oscillation. For the 3rd case, we have $x(t) = Ae^{-(b/2m)t}\cos(\omega t - \delta)$ where $\omega = \sqrt{4mk - b^2}/2m$. This is applicable to pendulum with air resistance.

3) *Forced undamped oscillator*: We have $m\ddot{x} + kx = F_0\cos\alpha t$ whose solution is $x(t) = A\cos(\omega t - \delta) + \frac{F_0}{\omega^2-\alpha^2}\cdot\cos\alpha t$ where $\omega = \sqrt{k/m}$. If we know $x(0) = \dot{x}(0) = 0$, then $x(t) = A(t)\sin(\frac{\alpha+\omega}{2}t)$ where $A(t) = \frac{F_0}{\alpha^2-\omega^2}\sin(\frac{\alpha-\omega}{2}t)$. **Beating** means the faster signal $(\alpha+\omega)/2$ is modulated by the slower one $(\alpha-\omega)/2$, so we can only hear $A(t)$; **resonance** means when $\alpha = \omega$, then $A(t) = F_0 t/(2m\omega)$ and oscillation goes out of control.

4) *Forced damped oscillator*: We have $m\ddot{x} + b\dot{x} + kx = F_0\cos\alpha t$. If t is big enough, steady state solution is $x(t) = x_p(t) = A(\alpha)\cdot\cos(\alpha t - \gamma)$ where $A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2-\alpha^2)^2+(b^2\alpha^2)/m^2}}$ and $\omega = \sqrt{k/m}$.

5) *Equilibrium & Stability*: Equilibrium solution means $x(t)$ is a constant or $\dot{x} = 0$. Equilibrium points are stable if points nearby stay close to it.

2. Buoyancy Force & Cantilevered Beam

1) *Archimedes' principle*: We have $m\ddot{x} = -xA\rho g$, which is similar to simple harmonic motion $\ddot{x} + \omega^2 x = 0$ where $\omega = \sqrt{A\rho g/m}$.

2) *Euler's equation*: We have $\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = W(x)$, which usually becomes $\frac{d^4y}{dx^4} = \frac{-\alpha}{EI}$ when $W(x)$ is constant. Then, the solution becomes $y(x) = \frac{\alpha L^4}{2EI}\left(-\frac{1}{12}\left(\frac{x}{L}\right)^4 + \frac{1}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right)$.

3. Population Model

0) *No-crossing principle*: Since there is exactly one solution for any 1st order ODE with given initial condition, curves never intersect.

1) *Malthus model*: Given $\frac{dN}{dt} = (B - D)\cdot N$, its solution is $N(t) = N_0e^{kt}$ where $k = B - D$. If $k > 0$, population explosion; if $k = 0$, population stable; if $k < 0$, population extinction.

2) *Logistic model*: Given $\frac{dN}{dt} = (B - sN)\cdot N$, its solution is $N(t) = \frac{B}{s+(\frac{B}{N_0}-s)e^{-Bt}}$ with sustainable value B/s. If $N_0 < B/s$, then $N(t)$ keeps increasing and tends to the sustainable value; if $N_0 > B/s$, then $N(t)$ keeps decreasing and tends to the sustainable value.

3) *Harvesting model*: Given $\frac{dN}{dt} = BN - sN^2 - E$, let $\Delta = B^2 - 4sE$. If $\Delta < 0$, no equilibrium point, keep decreasing until extinction; if $\Delta > 0$, have two guidelines (upper stable & lower unstable) and three regions, the upper region is good, the middle one is able to bounce back, the lower one is dangerous; if $\Delta = 0$, have one guideline and two increasing regions.

Extinction time: Integrate $T = \int_{N_0}^{0}\frac{1}{-sN^2+BN-E}dN$.

Part 2 Laplace Transform

1. Laplace transform for basic functions:

$$L(e^{at}) = \frac{1}{s-a} \qquad L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(\sin\omega t) = \frac{\omega}{s^2+\omega^2} \qquad L(\cos\omega t) = \frac{s}{s^2+\omega^2}$$

$$L(\sinh\omega t) = \frac{\omega}{s^2-\omega^2} \qquad L(\cosh\omega t) = \frac{s}{s^2-\omega^2}$$

2. Derivative transform:

$$L(f') = sL(f) - f(0) \qquad L(f'') = s^2L(f) - sf(0) - f'(0)$$

3. s-shifting: $L\left(e^{at}f(t)\right) = F(s-a)$ $L\left(t^nf(t)\right) = (-1)^nF^{(n)}(s)$

4. t-shifting: $L\left(f(t-a)u(t-a)\right) = e^{-as}F(s)$

5. Dirac delta function: We have $\delta(t - t_0) = 0$ for $t \neq t_0$ and $\int_{-\infty}^{\infty}\delta(t - t_0)dt = 1$, then $L\left(\delta(t - t_0)\right) = e^{-st_0}$.

6. Handle non-standard data: method of undetermined coefficients or method of function translation.

Part 3 Matrix

1. *$n\times n$ matrix*: symmetric $A^T = A$, anti-symmetric $A^T = -A$, identity $AI = IA = A$, orthogonal $B \cdot B^T = I$, involutory $AA = I$.
2. *Rotation matrix* (anti-clockwise): $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
3. *Shear matrix* (clockwise, parallel to x-axis): $\begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$.
4. *Matrix with solution of equations*: Given $A \cdot X = B$, so we have $X = A^{-1} \cdot B$, when A is a $n\times n$ matrix and B is a $n\times 1$ matrix.

When $X \neq 0$, a unique solution exists if $\det(A) \neq 0$. When $X = 0$, there are infinite non-zero solutions if $\det(A) = 0$, there is only a zero solution if $\det(A) \neq 0$.

5. *Leontief input-output model*: $X = (I - M)^{-1} \cdot D$.
6. *Eigenvalue & eigenvector*: Given $T\vec{u} = \lambda\vec{u}$, eigenvalues and corresponding eigenvectors can be found by $\det(T - \lambda I) = 0$.

Sum of the eigenvalues is the trace of the matrix, while product of the eigenvalues is the determinant of the matrix.

7. *Diagonalization*: For an $n\times n$ matrix A, it is diagonalizable if $A = PDP^{-1}$, where P is matrix of n non-parallel eigenvectors and D is the diagonal matrix of n eigenvalues.
8. *Weather forecast model*: we have $M^n = P\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}P^{-1}$.
9. *Markov chain*: For a transform M, becomes M^n after n times.
9. *Dimension of linear transformation*: An $m\times n$ matrix will transform a n-D vector into a m-D vector.
10. *Volume and determinant*: The determinant of a 2×2 matrix is the area of parallelogram, while that of a 3×3 matrix is the volume of parallelepiped (rank 3, rank 2, rank 1). For a plane generated by $\vec{u}$ and $\vec{v}$, $\vec{w}$ is on the plane if and only if $(\vec{u}\times\vec{v}) \cdot \vec{w} = 0$.

Part 4 System of ODEs

1. Method of eigenvalue: $\alpha = (Tr(B) \pm \sqrt{Tr^2(B) - 4\det(B)})/2$.

If $\Delta > 0$, the general solution is $c_1 u_1 e^{\alpha_1 t} + c_2 u_2 e^{\alpha_2 t}$ with eigenvalues α_1, α_2 and corresponding eigenvectors u_1, u_2; if $\Delta < 0$, the general solution is $e^{\alpha t} \cdot (u\cos\beta t - v\sin\beta t) + ie^{\alpha t} \cdot (u\sin\beta t + v\cos\beta t)$ where eigen-values are $\alpha \pm i\beta$.

2. Method of Laplace transform: Let $\vec{v}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, then solve $\overrightarrow{v'}(t) = B\vec{v}$. We have $L(\vec{v}) = (sI - B)^{-1} \cdot \vec{v}(0)$.
3. Phase plane and classification of zero solution:

1) *When* $\Delta > 0$, *two roots*: nodal source – unstable, $\alpha_1, \alpha_2 > 0, TrB > 0, detB > 0$; nodal sink – stable, $\alpha_1, \alpha_2 < 0, TrB < 0, detB > 0$; saddle point – unstable, $\alpha_1\alpha_2 < 0, detB < 0$.

2) *When* $\Delta < 0$, *two complex roots*: spiral source – unstable, $TrB > 0$; spiral sink – stable, $TrB < 0$;center – stable, $TrB = 0$.

3) *To determine clockwise or anti-clockwise*: Let $y = 0$, check the sign of dy/dt near positive x-axis.

4. Warfare model: Compare with the gradient of basic trajectory lines to decide which side will win.

Part 5 Partial Differential Equations (PDEs)

1. Sturm-Liouville equation: For $X''(x) + \lambda X(x) = 0$, non-zero solution $X(x) = C\sin\left(\frac{n\pi}{L}x\right)$ exists if and only if $\lambda = \left(\frac{n\pi}{L}\right)^2$.
2. Wave equation: For $c^2 y_{xx}(x,t) = y_{tt}(x,t)$, given $y(0,t) = y(L,t) = 0$ and $y(x,0) = f(x), y_t(x,0) = 0$, the solution is $y(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x)\cos\left(c\frac{n\pi}{L}t\right)$, where $A_n = \frac{2}{L}\int_0^L f(x) \cdot \sin(\frac{n\pi}{L}x)\,dx$. Or, you can directly *compare the coefficients*.

d'Alembert's solution: $y(x,t) = \big(f(x+ct) + f(x-ct)\big)/2$.

3. Heat equation: For $u_t(x,t) = c^2 u_{xx}(x,t)$, given $u(x,0) = f(x)$ and $u(0,t) = u(L,t) = 0$, the solution is $u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x) \cdot e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}$ where $A_n = \frac{2}{L}\int_0^L f(x) \cdot \sin(\frac{n\pi}{L}x)\,dx$.
4. Laplace equation: For $u_{yy}(x,y) = -u_{xx}(x,y)$, given $u(x,0) = f(x), u(x,K) = 0$ and $u(0,y) = u(L,y) = 0$, then the solution is $u(x,y) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x)\sinh\left(\frac{n\pi c}{L}(y-k)\right)$ where we have $A_n = \frac{2}{L\sinh\frac{-n\pi cK}{L}}\int_0^L f(x)\sin(\frac{n\pi}{L}x)\,dx$.
5. Useful Integral: $\int x\sin\lambda x\,dx = \frac{\sin\lambda x}{\lambda^2} - \frac{x\cos\lambda x}{\lambda}$

$$\int x^2\sin\lambda x\,dx = \frac{2x\sin\lambda x}{\lambda^2} + \frac{(2-\lambda^2x^2)\cos\lambda x}{\lambda^3}$$

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