MA1506 Midterm Cheat-sheet
AY2016/2017 Semester 2

Part 1 Solving Ordinary Differential Equations (ODEs)

You should use the following methods to solve ODEs in the exact order, from the first one to the last one.

1. Separable equations

1) Directly separable: Puts all x terms at the right-hand side and all y terms at the left-hand side. Integrate both sides together. Notice that the constant c should be placed at the right-hand side and determined by initial condition.

2) Method of substitution

By observation, decide how to substitute y into some combination of x and y, such as $\frac{y}{x}$ and $ax + by$.

2. Linear 1st order ODEs

1) Integrating factor method

For standard form equation like $\frac{dy}{dx} + P(x)y = Q(x)$, we know the integrating factor is $e^{\int P(x)dx}$. Therefore, the solution to such an equation can be found by $y \cdot e^{\int P(x)dx} = \int Q(x) \cdot e^{\int P(x)dx}\, dx$.

2) Bernoulli equation

For equations of the form $y' + P(x)y = Q(x)y^n$, we will substitute $z = y^{1-n}$ to transform them into linear 1st order ODEs.

3. Homogenous linear 2nd order ODEs

1) Make sure the equation is in the form of $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$.

2) Obtain the characteristic equation $\lambda^2 + A\lambda + B = 0$.

3) Two distinct real roots: $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

Vieta's formulas: $\lambda_1 + \lambda_2 = -\frac{b}{a}$ and $\lambda_1\lambda_2 = \frac{c}{a}$

4) One real root: $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$.

5) Two distinct complex roots: $y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$.

4. Non-homogenous linear 2nd order ODEs

1) Method of undetermined coefficients

For equations of the form $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = R(x)$, we know the particular solution to the equation would be

a. Only the following 5 types can be solved by this method:

S/N	R(x)	y_p
1	C	$x^s A$
2	$P_n(x)$	$x^s Q_n(x)$
3	$\beta \cdot e^{\alpha x}$	$x^s A e^{\alpha x}$
4	$\beta \cdot \sin/\cos bx$	$x^s(A \sin bx + B \cos bx)$
5	$\beta x e^{\alpha x}$	$x^s(Ax + B)e^{\alpha x}$

(where $s = 0, 1, 2$)

b. Rule of increasing order

In the above form, we define s as the smallest non-negative integer such that no term in y_p is a solution or part of the solution of the corresponding homogenous (complex-value) ODE.

c. Method of complex-value ODEs

When $y_p = x\big((Ax + B)\sin bx + (Cx + D)\cos bx\big)$, we should use complex-value ODEs to help us find the particular solution.

Since we know $e^{bix} = \cos bx + i \sin bx$, we transform the original ODE $y'' + Ay' + By = \beta \sin/\cos bx$ into its corresponding complex value ODE $z'' + Az' + Bz = \beta e^{bix}$. Solve this complex-value ODE and y will be the real/imaginary part of z.

2) Method of variation of parameters

a. Formula: For a certain 2nd order ODE, if $y_h = c_1 y_1(x) + c_2 y_2(x)$, then we know $y_p = u(x) \cdot y_1(x) + v(x) \cdot y_2(x)$, where

$$u(x) = \int \frac{-r \cdot y_2}{y_1 y_2' - y_1' y_2} dx \qquad v(x) = \int \frac{r \cdot y_1}{y_1 y_2' - y_1' y_2} dx$$

b. *Caution*: Avoid using method of variation of parameters as much as possible. Usually, only the last question need it.

3) Superstition principle

If the right-hand side of a non-homogenous 2nd order can be separated into a few linear-independent terms, we can solve them as isolated 2nd order ODEs and combine them after that.

Good Luck!

Part 2 Applications of ODEs

1. Radioactive Decay Problem

The formula is $x(t) = x(0) \cdot e^{-\frac{\ln 2}{\tau}t}$, where τ is the half-life.

2. Heat Transfer Problem

By Newton's Law of Cooling (Rate of change of temperature is proportional to the temperature difference), the formula is $T(t) = T_0 + (T(0) - T_0)e^{kt}$, where T_0 is the temperature of the environment and $T(0)$ is the initial temperature.

3. Mixture Problem

By definition of concentration, we know $\frac{dx}{dt} = A \cdot c - \frac{x}{V} \cdot A$, where V is the volume of the container, A is the flowing rate and c is the concentration from the flow-in fluid. Use integrating factor method to solve the equation and get the answer.

4. Retarded Fall – Air Resistance

1) Linear relationship

The air resistance is proportional to the velocity, by Newton's 2nd Law, we have $m\frac{dv}{dt} = mg - bv$, where g is the gravitational constant. Use integrating factor method to solve the equation and get the answer. In fact, this equation is also separable.

2) Square relationship

The air resistance is proportional to the square of the velocity, by Newton's 2nd Law, we have A $m\frac{dv}{dt} = mg - bv^2$, , where g is the gravitational constant. This equation is separable.

5. Decay Chain Problem

Assuming that substance A converts to substance B of the same quantity, and they both follow the formula in the radioactive decay problem. We have A $\frac{B(t)}{A(t)} = \frac{K_A}{K_B - K_A} \cdot (1 - e^{(K_A - K_B)t})$, where $K_{A/B}$ is the decay constant of that substance.

Part 3 Miscellaneous

1. Commonly-used Differentials

1) Polynomial functions:

$$(A \cdot B)' = A'B + AB' \qquad \left(\frac{A}{B}\right)' = \frac{A'B - AB'}{B^2}$$

2) Trigonometric functions:

$$(\sin x)' = \cos x \quad (\cos x)' = -\sin x \quad (\tan x)' = \sec^2 x$$

$$(\sec x)' = \sec x \cdot \tan x \quad (\csc x)' = -\csc x \cdot \cot x \quad (\cot x)' = -\csc^2 x$$

3) Inverse trigonometric functions:

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \quad (\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}} \quad (\tan^{-1} x)' = \frac{1}{1+x^2}$$

4) Exponential and logarithmic functions:

$$(e^x)' = e^x \quad (a^x)' = a^x \cdot \ln a \quad (\ln x)' = \frac{1}{x}$$

2. Commonly-used Integrals

1) Polynomial functions:

$$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + C \quad \int \frac{ax+b}{cx+d} dx = \frac{a}{c}x + \frac{bc-ad}{c^2}\ln(cx+d) + C$$

$$\int \frac{1}{x^2-k^2} dx = \frac{1}{2k} \cdot \ln\frac{x-k}{x+k} + C \quad \int \frac{1}{x^2+k^2} dx = \frac{1}{k} \cdot \tan^{-1}\frac{x}{k} + C$$

2) Trigonometric functions:

$$\int \sin x\, dx = -\cos x + C \quad \int \cos x\, dx = \sin x + C$$

$$\int \tan x\, dx = \ln \sec x + C \quad \int \cot x\, dx = \ln|\sin x| + C$$

$$\int \sec^2 x\, dx = \tan x + C \quad \int \csc^2 x\, dx = -\cot x + C$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

3) Exponential and logarithmic functions:

$$\int x^n e^{-x}\, dx = \int -x^n\, de^{-x} \quad \text{(Integration by parts)}$$

3. Hyperbolic Functions

1) Definition:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

2) Differentials:

$$(\sinh x)' = \cosh x \qquad (\cosh x)' = \sinh x$$

$$(\tanh x)' = 1 - tanh^2 x = \operatorname{sech}^2 x$$

3) Integrals:

$$\int \sinh cx\, dx = \frac{1}{c}\cosh cx + C \quad \int \cosh cx\, dx = \frac{1}{c}\sinh cx + C$$

$$\int \tanh cx\, dx = \frac{1}{c}\ln(\cosh cx) + C$$

Good Luck!